

Simplified procedures for calculation of instantaneous and long-term deflections of reinforced concrete beams

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ABSTRACT

The simplified methods of CEB and of ACI for prediction of deflections of reinforced concrete beams are analyzed in this work. Obtained results with those methods are compared with those obtained through a nonlinear analysis. Additional deflections due to creep and shrinkage are considered in the study. The obtained results indicate that the two simplified methods are appropriate for calculation of instantaneous deflections of reinforced concrete beams. The method of CEB also supplies good results when creep and shrinkage of concrete are considered. However, the method of ACI is not satisfactory for evaluation of long-term deflections of reinforced concrete beams.

1. INTRODUCTION

Actual design procedures of reinforced concrete structures are based in concepts of limit states. They are generally classified as ultimate limit states and serviceability limit states. The ultimate limit state refers to the bearing capacity of a structural part or of the structure as a whole. The serviceability limit states are associated with the structural performance under expected actions in normal conditions of utilization.

In the verifications of the serviceability limit states, actions are considered with characteristic values, which is equivalent to consider partial safety factors equal to one. In the same way, mean mechanical properties of the materials are used to evaluate the structural rigidity.

Usually, design codes for reinforced concrete structures require that beam deflections are calculated for the quasi-permanent combination of the actions. In each combination, permanent action is represented by a single representative value, g . Usually, this value is considered as a mean value, which is calculated from nominal dimensions of the structural elements. Variable actions are considered with the quasi-permanent value $\psi_2 q$, where q is the characteristic value and $\psi_2 < 1$. This is the format of CEB-FIP Model Code (1993), for example.

According to the Brazilian code NBR-8681 (ABNT(2003)), for the accidental loads of residential buildings and of office buildings, it is necessary to consider $\psi_2 = 0.3$. In these cases, the quasi-permanent load p_o is given by

$$p_o = g + 0.3q \quad (1)$$

where g and q are the characteristic values of the permanent load and the accidental load, respectively.

In a consistent analysis of reinforced concrete beam, it is necessary to take into account the cracking and the tension stiffening effect. Creep and shrinkage of concrete should also be considered for calculation of long-term deflection.

The analysis may be accomplished with different refinement levels, from the nonlinear analysis, until the utilization of simplified formulas to represent an equivalent rigidity of the cracked beam, as in the simplified method adopted in ACI Code (1995). This method of ACI has been adopted in many national codes, as in EHE (1999) of Spain, in NBR-6118 [ABNT(2004)] of Brazil, and in the codes of several countries of Latin America.

On the other hand, the CEB (1985) recommends the use of the bilinear method, which is adopted in Eurocode 2 (2003) with small modifications.

The objective of this work is to verify the precision of those two simplified methods, which will be named method of ACI and method of CEB. The nonlinear model presented in the following section is considered as the reference model.

2. NONLINEAR MODEL FOR REINFORCED CONCRETE BEAMS ANALYSIS

The nonlinear model used in this work had its precision demonstrated in previous works [Araújo (1991) and Araújo (2003)] and it allows obtaining the probable deflections of reinforced concrete beams.

The stress-strain diagram for concrete subjected to compression is shown in Fig. 1.

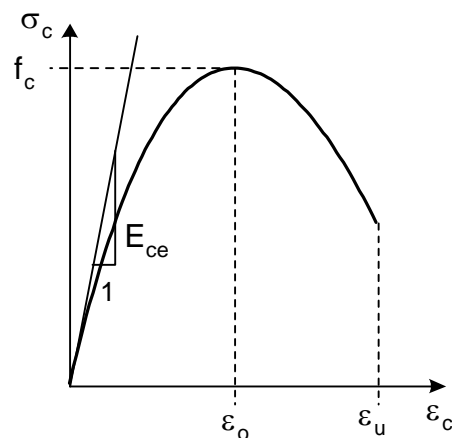


Fig. 1 – Stress-strain diagram for concrete in uniaxial compression

According to CEB (1993), the compression stress σ_c , is given by

$$\sigma_c = -f_c \left[\frac{k\eta - \eta^2}{1 + (k-2)\eta} \right] \quad (2)$$

where $k = -E_{ce}\epsilon_o/f_c$; $\eta = \epsilon_c/\epsilon_o$ and ϵ_c is the compression strain.

For the strain ϵ_o , corresponding to the peak compressive stress f_c , the value $\epsilon_o = -0.0022(1 + \varphi)$ is adopted, where φ is the creep coefficient. The ultimate strain is $\epsilon_u = -0.0035(1 + \varphi)$.

The initial modulus of elasticity of concrete, E_c , may be estimated from the mean compressive strength, f_c , by means of relation

$$E_c = 21500 (f_c/10)^{1/3} \quad (3)$$

where f_c and E_c are given in MPa.

According to CEB (1993), the mean strength for use in expression (3) may be estimated as

$$f_c = f_{ck} + 8 \text{ MPa} \quad (4)$$

where f_{ck} is the characteristic compressive strength of concrete, given in MPa.

For long-term loading, the total deformation including creep is calculated by using an effective modulus of elasticity for concrete, E_{ce} , given by

$$E_{ce} = E_c / (1 + \varphi) \quad (5)$$

For concrete subjected to tension are adopted the stress-strain diagrams indicated in Fig. 2.

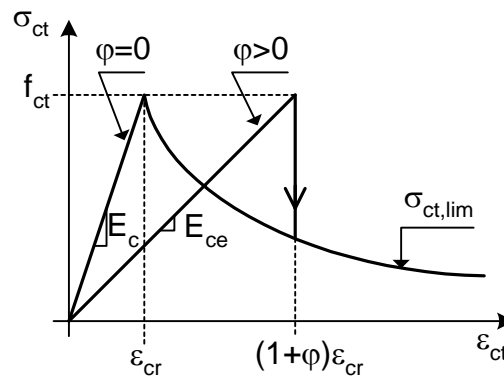


Fig. 2 – Stress-strain diagrams for concrete in tension

The tensile stress σ_{ct} is given by

$$\sigma_{ct} = \begin{cases} E_{ce} \varepsilon_{ct}, & \text{se } \varepsilon_{ct} \leq (1 + \varphi) \varepsilon_{cr} \\ \sigma_{ct,lim}, & \text{se } \varepsilon_{ct} > (1 + \varphi) \varepsilon_{cr} \end{cases} \quad (6)$$

where ε_{cr} is the tensile strain and $\sigma_{ct,lim}$ is the maximum tensile stress for cracked concrete, given by

$$\sigma_{ct,lim} = f_{ct} \left(\frac{\varepsilon_{cr}}{\varepsilon_{ct}} \right)^{0.6} \quad (7)$$

where $\varepsilon_{cr} = f_{ct}/E_c$ is the cracking strain of concrete for short-term loads.

The expression (7) takes into account the tension stiffening effects and reproduces the experimental results satisfactorily, as it may be verified in Araújo (1991).

According to CEB (1993), axial tensile strength f_{ct} may be estimated through the expression

$$f_{ct} = 1,40 \left(\frac{f_{ck}}{10} \right)^{2/3} \quad (8)$$

with f_{ck} and f_{ct} given in MPa.

As it may be observed, the constitutive model is an extension of the effective modulus method for the nonlinear case. This model is satisfactory when the history of stress is characterized by limited variations during the aging of concrete. When great stress variations occur during the aging period, the model may be improved applying the adjusted effective modulus method, as in Ghali and Favre (1986) and in CEB (1984). That is equivalent to consider an adjusted creep coefficient $\varphi_a = \zeta\varphi$, where ζ is the aging coefficient. In the practical applications $\zeta = 0.8$ may be adopted.

In this work $\zeta = 1$ is adopted because it is admitted that the quasi-permanent load p_o doesn't vary along the time.

The total strain of concrete, $\varepsilon_{c,tot}$, is given by

$$\varepsilon_{c,tot} = \varepsilon_{c\sigma} + \varepsilon_{cs} \quad (9)$$

where $\varepsilon_{c\sigma}$ is the stress dependent strain and ε_{cs} is the shrinkage strain.

Therefore, the stresses in the concrete may be obtained with the previous model considering the portion of the strain $\varepsilon_{c\sigma} = \varepsilon_{c,tot} - \varepsilon_{cs}$.

The structural analysis is made by means of the finite element method. The finite element used is the classic element for plane frames, with two nodes and three degrees of freedom for node. Three Gauss integration points are considered along the finite element length, for determination of the nodal nonlinear actions.

At each integration point the concrete cross-section is discretized in $n = 30$ layers in the height direction. The resistant sectional forces are obtained considering the stresses in the reinforcement and in the center of each concrete layer [Araújo (1991)].

Using the finite element method, the strain ε_x in a generic layer of the cross-section, located to a distance z from centroidal axis of the gross section, neglecting reinforcement, is given by

$$\varepsilon_x = \varepsilon_{x0} + z\chi \quad (10)$$

where ε_{x0} and χ represent the axial strain and the curvature, obtained through the nodal displacements of the element.

The expression (10) supplies the total strains in the concrete, including creep and shrinkage. The mechanical strain $\varepsilon_{c\sigma}$ is given by

$$\varepsilon_{c\sigma} = \varepsilon_{x0} + z\chi - \varepsilon_{cs} \quad (11)$$

where $z = z_i$, with $i = 1, \dots, n$, represents the distance of the center of concrete layer until the centroidal axis of gross section.

After the calculation of the strain $\varepsilon_{c\sigma}$, the constitutive model is used to obtain the stress in the center of each concrete layer.

The expression (10) is also used to calculate the strain in each steel layer, being enough to use for z the distance of the steel layer to the centroidal axis of gross section. It is

assumed that the reinforcing steel presents a perfect elastic-plastic behavior in tension and in compression.

Using the finite element method it is obtained the displacements of the structure at a certain load level. The nonlinear equations system, due to material nonlinearities, is solved iteratively using the BFGS method. Small load increments are applied on the beam until the rupture occurs.

Occurrence of the rupture is admitted when the compression strain in the concrete results smaller or equal the $\varepsilon_u = -0.0035(1 + \varphi)$, or when the tensile strain in the steel reaches the value limits of 0.010.

3. SIMPLIFIED PROCEDURE OF ACI

According to ACI Code (1995), instantaneous deflections of reinforced concrete beams, $W(t_o)$, shall be computed with the effective moment of inertia I_e , given by

$$I_e = \left(\frac{M_r}{M}\right)^3 I_c + \left[1 - \left(\frac{M_r}{M}\right)^3\right] I_2 \leq I_c \quad (12)$$

where I_c = moment of inertia of gross concrete section about centroidal axis, neglecting reinforcement; I_2 = moment of inertia of cracked section transformed to concrete; M = maximum moment in member at stage deflection is computed; M_r = cracking moment.

The cracking moment is given by

$$M_r = \frac{I_c f_{ct}}{y_t} \quad (13)$$

where y_t is the distance from centroidal axis of the gross section, neglecting reinforcement, to extreme fiber in tension.

Thus, for a rectangular section result the expression $M_r = bh^2 f_{ct} / 6$.

For prismatic members, effective moment of inertia I_e may be calculated considering the geometrical properties of the critical section.

ACI Code adopted different expressions for the modulus of elasticity and for the tensile strength of concrete. However, E_c and f_{ct} will be calculated through the expressions (3) and (8) for those properties to be the same in all the analyzed methods.

To consider possible material nonlinearities, rigidity is evaluated with the secant modulus of elasticity $E_{cs} = 0.85E_c$. Thus, deflections are calculated considering the effective rigidity given by $E_{cs}I_e$.

Additional long-term deflection, ΔW , resulting from creep and shrinkage, is obtained by

$$\Delta W = \frac{\xi}{1 + 50\rho'} W(t_o) \quad (14)$$

where $W(t_o)$ is the instantaneous deflection caused by the sustained load and $\rho' = A'_s / (bd)$ is the compression reinforcement ratio on the critical section.

Critical section may be taken at midspan for simple and continuous beams, and at support for cantilevers.

The factor ξ depends on the duration of the load, being $\xi = 2.0$, for 5 years or more, and $\xi = 1.4$, for 12 months of duration of the load.

The total deflection of the beam, W , is given by

$$W = W(t_o) + \Delta W \quad (15)$$

4. THE BILINEAR METHOD OF CEB

In the bilinear method (CEB (1985)), the deflection due to load, including creep and shrinkage, is given by

$$W = (1 - \eta)W_1 + \eta W_2 \quad (16)$$

where W_1 and W_2 are the deflections calculated for uncracked and fully cracked conditions, respectively.

In order to calculate W_1 , it is considered the moment of inertia I_1 of uncracked section transformed to concrete. For to obtain W_2 , it is considered the moment of inertia I_2 of cracked section transformed to concrete. Critical sections are defined as previously.

Coefficient η , allowing for tension stiffening effect, is given by

$$\eta = 0, \quad \text{if } M < M_r \quad (17)$$

$$\eta = 1 - \beta \frac{M_r}{M}, \quad \text{if } M \geq M_r \quad (18)$$

where $\beta = 1.0$ for a single short-time loading and $\beta = 0.5$ for sustained loads or many cycles of repeated loading.

Creep may be included by using the effective modulus of elasticity for concrete according to expression (5). To consider material nonlinearities, it is adopted the effective secant modulus of elasticity $E_{cse} = 0.85E_{ce}$.

Shrinkage curvatures $\chi_{cs,1}$, for uncracked sections, and $\chi_{cs,2}$, for cracked sections, may be assessed by expressions

$$\chi_{cs,1} = \varepsilon_{cs} \alpha_e \frac{S_1}{I_1} ; \quad \chi_{cs,2} = \varepsilon_{cs} \alpha_e \frac{S_2}{I_2} \quad (19)$$

where ε_{cs} is the free shrinkage strain; I_1 and I_2 are the moments of inertia of uncracked and cracked section transformed to concrete, respectively; S_1 and S_2 are the first moments of area of the reinforcement about centroid of the transformed section in the uncracked and cracked stages; $\alpha_e = E_s/E_{cse}$ is the effective modular ratio.

Thus, additional deflections $W_{cs,1}$ and $W_{cs,2}$ resulting from shrinkage are obtained by integration of the curvatures given in expression (19). For a single span beam, it results

$$W_{cs,1} = \frac{L^2}{8} \chi_{cs,1} \quad ; \quad W_{cs,2} = \frac{L^2}{8} \chi_{cs,2} \quad (20)$$

where L is the span of the beam.

Finally, $W_{cs,1}$ and $W_{cs,2}$ are used in the expression (16) to obtain the additional deflection caused by shrinkage.

5. EXAMPLES

The reinforced concrete single span beam of constant cross-section shown in Fig. 3 is analyzed in this work. The uniform loading is composed by the permanent load g and by the accidental load q . Assuming that $q \cong 0.15g$ and considering the equation (1), it results $p_o = 0.90p_k$, where $p_k = g + q$ is the total service load on the beam.

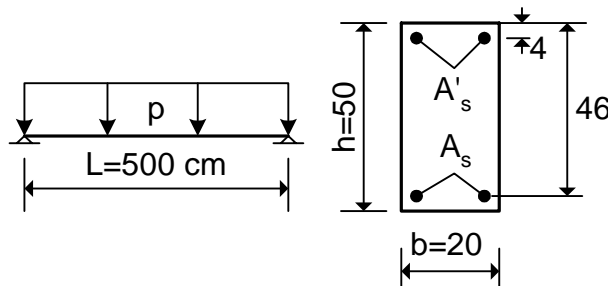


Fig. 3 – Loading and geometry of the beam

In the analyzed examples, it is admitted that the characteristic compressive strength of concrete is $f_{ck} = 20$ MPa. Using the expressions (3) and (4), it is obtained the initial modulus of elasticity of concrete $E_c = 30300$ MPa. The secant modulus of elasticity is $E_{cs} = 0.85E_c = 25755$ MPa. Through the expression (8), it is obtained the tensile strength of concrete $f_{ct} = 2.22$ MPa. The creep coefficient is considered equal the $\varphi = 2.5$ and free shrinkage strain is $\varepsilon_{cs} = -50 \times 10^{-5}$.

The characteristic yield stress of reinforcing steel is $f_{yk} = 500$ MPa and it is assumed a modulus of elasticity $E_s = 200$ GPa.

Service loads $p_k = 10$ kN/m, $p_k = 15$ kN/m and $p_k = 20$ kN/m are considered for study. Those values represent the service loads that usually act in the beams of the residential buildings.

The steel areas in the beam cross-section are calculated considering partial safety factors given in NBR-6118 [ABNT(2004)]. This calculation indicates that $A'_s = 0$ for the three cases of loading. Then, $A'_s = 0.62 \text{ cm}^2$ is adopted as compressive steel. Table 1 indicates the loads and steel areas of the three analyzed beams.

Beam	Load (kN/m)		Steel area (cm ²)	
	p_k	p_o	A_s	A'_s
B1	10	9.0	2.29	0.62
B2	15	13.5	3.52	0.62
B3	20	18.0	4.83	0.62

6. RESULTS

The answers of the three beams, obtained with the nonlinear model, with the bilinear method of CEB and with the method of ACI, are shown in figures 4 to 6. These figures indicate the relationships between the uniform load and the midspan instantaneous deflection.

It may be observed that there is a good agreement among the three methods in all the stages of the loading. In the proximities of the cracking load, the simplified methods of CEB and ACI supply a larger initial deflection than the nonlinear model. This happens because the steel areas were not included in the calculation of the cracking moment in those simplified methods.

A better adjustment may be obtained, for this loading stage, if the steel areas are included in the calculation of the cracking moment M_r . However, this stage is not of larger importance, because the quasi-permanent load is usually larger than the cracking load, as it is observed in figures 4 to 6.

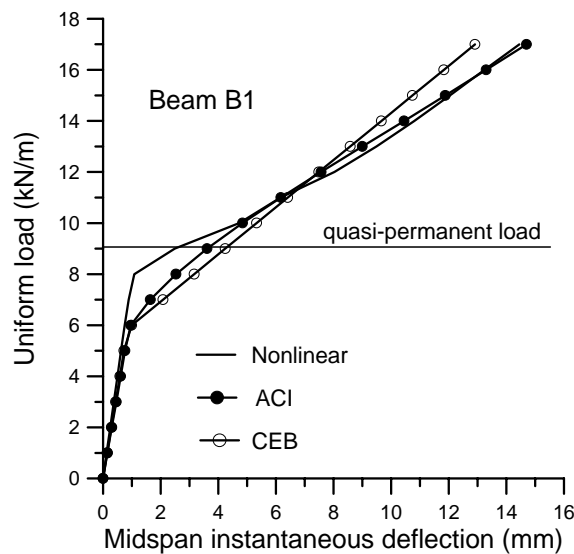


Fig. 4 – Curves load-instantaneous deflection for beam B1

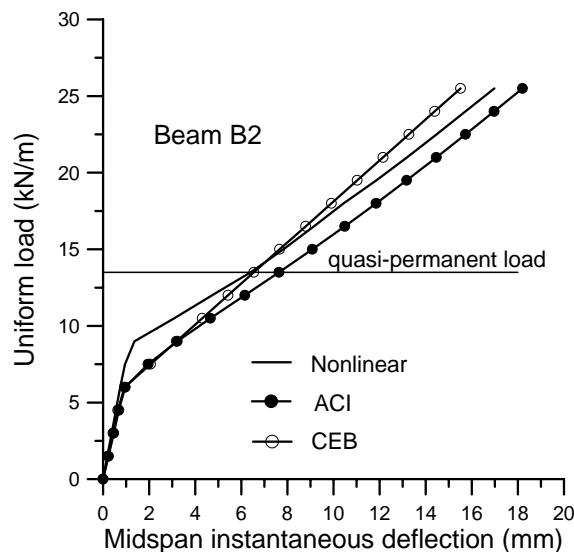


Fig. 5 – Curves load-instantaneous deflection for beam B2

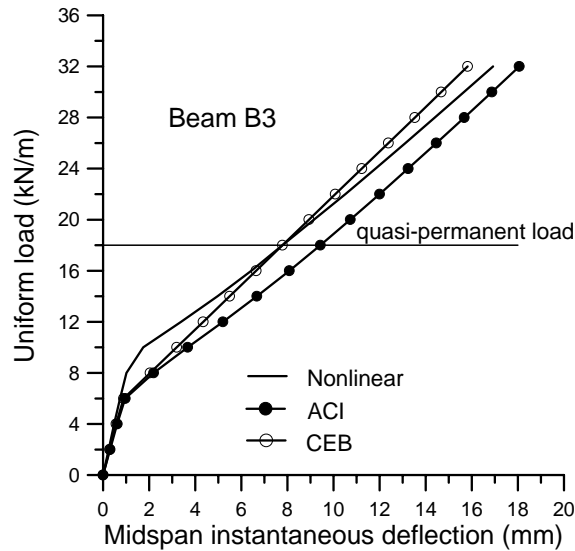


Fig. 6 – Curves load-instantaneous deflection for beam B3

Table 2 indicates the values of the instantaneous deflection for the quasi-permanent load p_o . As it is observed, the two simplified methods agree satisfactorily with the nonlinear analysis.

Table 2 – Instantaneous deflection (mm) for the quasi-permanent load

Beam	Method		
	Nonlinear	CEB	ACI
B1	2.5	4.2	3.6
B2	6.4	6.5	7.6
B3	7.8	7.8	9.4

Figures 7 to 9 show the relationships between the load and the midspan total deflection, including creep and shrinkage effects. For the method of ACI, coefficients $\xi = 1.4$ and $\xi = 2.0$ are considered.

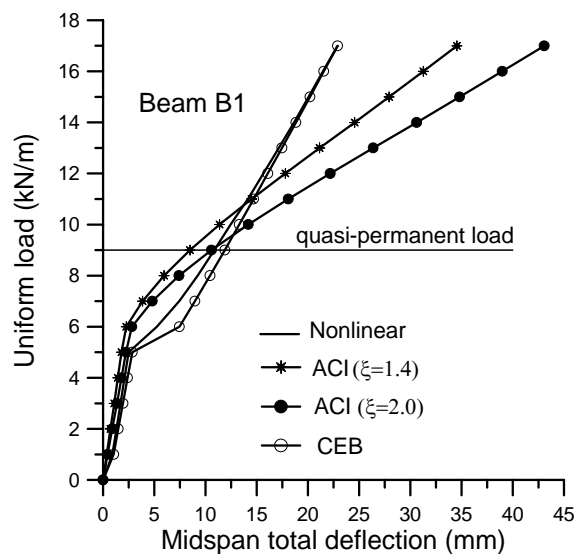


Fig. 7 - Curves load-total deflection for beam B1

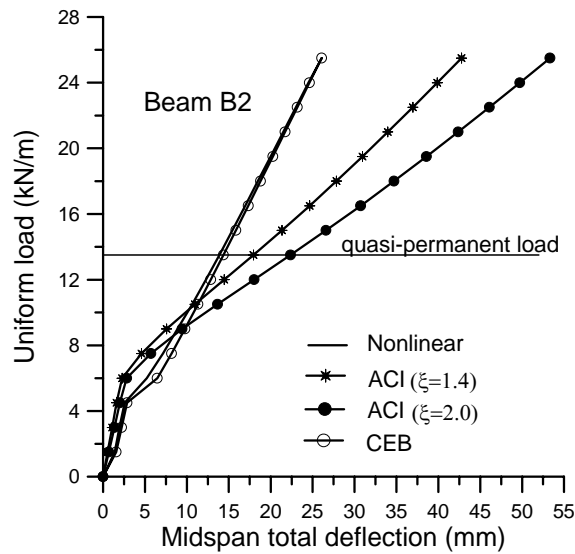


Fig. 8 - Curves load-total deflection for beam B2

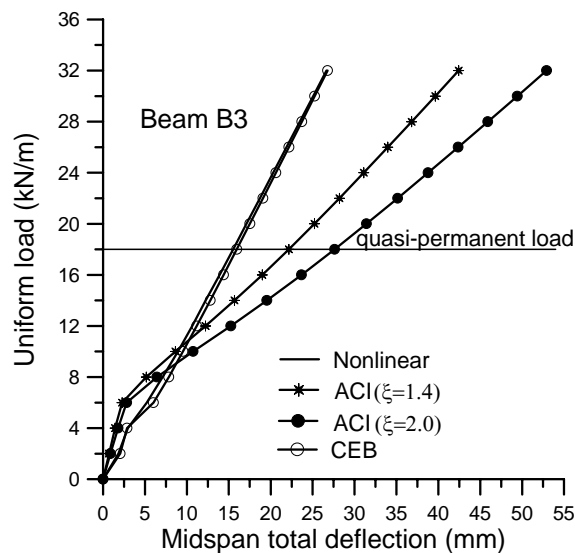


Fig. 9 - Curves load-total deflection for beam B3

As it is observed in figures 7 to 9, the bilinear method of CEB agrees very well with the nonlinear model for all the analyzed beams. That good agreement is verified for all the levels of loading.

However, the simplified method of ACI diverges enough of the nonlinear model. When the load is small and the beam is in the uncracked state, the method of ACI underestimates the total deflection. On the other hand, this method overestimates the total deflections for higher loads. In a general way, the total deflection is overestimated for the quasi-permanent load.

Table 3 indicates the values of total deflection obtained with the three methods for the quasi-permanent load p_o . As it is observed, the method of ACI overestimates the total deflection, except for the beam B1 and for the coefficient $\xi = 2.0$.

Table 3 – Total deflection (mm) for the quasi-permanent load

Beam	Method			
	Nonlinear	CEB	ACI $\xi = 1.4$	ACI $\xi = 2.0$
B1	10.9	11.9	8.5	10.6
B2	13.9	14.4	17.9	22.4
B3	15.5	16.0	22.2	27.6

With the values of the tables 2 and 3, it can be obtained the relationships between the additional deflection ΔW and the instantaneous deflection $W(t_o)$. Those relationships are presented in the Fig. 10.

As it is observed, the relationship $\Delta W/W(t_o)$ depends on the loading level. Then, the expression (14) represents only a rude approach of creep and shrinkage effects in the reinforced concrete beams deflections.

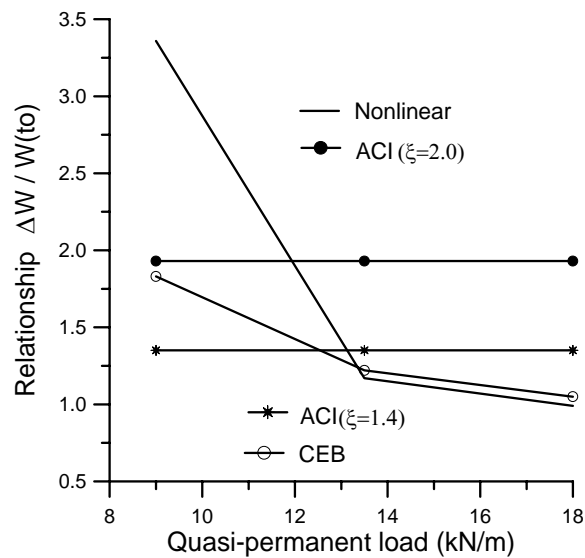


Fig. 10 - Variation of $\Delta W/W(t_o)$ as a function of the load

7. CONCLUSIONS

As a consequence of the results presented, it may be concluded that the bilinear method of CEB, as well as the method of ACI, are satisfactory for the evaluation of instantaneous deflections of reinforced concrete beams. Both methods may be used for beams in the uncracked state and in the cracked state, being obtained good results.

The bilinear method also supplies good results when creep and shrinkage effects are considered. This method may be used for calculation of the total displacements of beams, for several stages of the loading.

However, the method of ACI is not appropriate for evaluation of the total deflections of reinforced concrete beams. When this method is used, the following mistakes are expected:

- In structural elements that behave in an uncracked state, as solid slabs and beams submitted to loads of small intensity, the effects of the concrete delayed strains (creep and shrinkage) are underestimated. In this case, the design is not reliable in relation to the limit state of deformation.

- In elements that behave in the cracked state, as most of the beams of buildings, the effects of the concrete delayed strains are overestimated. In this case, the design is anti-economical.

Besides, the expression (14) is independent of the creep coefficient and of the shrinkage strain. That expression was determined empirically, based on a series of experimental results (Branson (1971), Yu and Winter (1960)). Consequently, it is only appropriate to reproduce the specific conditions adopted in those tests.

Any general expression for the relationship $\Delta W/W(t_o)$ should take into account the following factors:

- degree of cracking of the beam, measured through the relationship M_r/M ;
- steel rates $\rho = A_s/(bd)$ and $\rho' = A'_s/(bd)$;
- value of the creep coefficient φ ;
- value of the shrinkage strain ε_{cs} .

Consequently, the employment of the method of ACI is not recommended for calculation of deflections of reinforced concrete beams due to creep and shrinkage of concrete.

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